Inelastic impact of a sphere on a massive plane: nonmonotonic velocity-dependence of the restitution coefficient

Hunter King, Ryan White, Iva Maxwell,* and Narayanan Menon†
Department of Physics, University of Massachusetts, Amherst, MA 01003 U.S.A.

We have studied the coefficient of restitution, η, in normal collisions of a non-rotating sphere on a massive plate for a range of material parameters, impact velocity and sphere size. The measured coefficient of restitution does not monotonically vary with velocity. This effect is due to dynamics that occur during the finite duration of impact: the contact time varies as a function of velocity is comparable to the time-scales of the vibrational modes of the plate. The measured effect is robust and is expected to be ubiquitous in fluidized granular media. We also find that η is a decreasing function of particle size, a dependence that is not captured by existing models of impact.

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Collisions of macroscopic objects - such as a ball with the floor - are typically inelastic: some fraction of their total translational kinetic energy is siphoned off into viscoelastic work, plastic deformations, vibrations of the objects, and into producing sound. After careful experiments on normal collisions of spheres, Newton [1] suggested that the degree of inelasticity could be characterized by the ratio $\eta = -v\prime/v$, where v and $v\prime$ are the relative velocities before and after impact. The ratio η, called the coefficient of restitution, was at first thought to be a constant whose value was determined solely by the geometry and the material properties of the colliding objects. It is now well-known that η also depends on the relative velocity of impact: experiments as well as theoretical models [2, 3] indicate that $\eta \to 1$ as $v \to 0$ i.e. the gentler the impact, the closer it is to an elastic collision. In this article we study a particularly simple inelastic collision, that of a sphere colliding with a massive wall, and present data that show for the first time that η is nonmonotonic in v and is a decreasing function of the radius, R, of the sphere. The data suggest that computing $\eta(v)$ requires a fuller consideration of the dynamics of the colliding objects in the finite duration of the impact.

The starting point for most models of normal inelastic collisions is the Hertz solution to the static problem of a sphere that is being pushed into a wall [4]. This solution – which specifies the stress field in terms of compression of the sphere, its radius R and the elastic moduli of sphere and wall – is also assumed to obtain at any instant during a collision, under the condition that the velocity of impact v is much less than the speed of sound in the solids. To compute the coefficient of restitution, a model for what is judged to be the dominant dissipation mechanism supplements the Hertzian specification of the elastic force. In recent calculations [5, 6, 7], the dissipation mechanism has been modeled by a viscous damping term that is linear in the local strain rate. This yields the prediction $\eta_{visc}(v) \sim 1 - CR^{-1}v^{1/5}$ where C is a material-dependent constant. A different calculation that attributes the dissipation to plastic deformation [3, 8] predicts $\eta_{plastic}(v) \simeq 1.18(v/v_y)^{-1/4}$ for $v > v_y$, the velocity at which the yield stress is first exceeded.

Experiments on ball-ball and ball-plane collisions [2] generally show that η decreases with v, in qualitative agreement with the theoretical expectation. At high impact velocities the data are limited in range but moderately good agreement has been claimed [3] with a $v^{-1/4}$ dependence. At lower impact velocities the situation is less clear: the data of Labous et al. [9] for collisions between nylon beads are not fit very well by either $\eta_{plastic}$ or η_{visc}. The data of Hatzes et al. [10] on collisions of smooth ice spheres with ice bricks have been fit by $Cexp(-\gamma v)$. Falcon et al. [11] find η almost independent of v for collisions of a carbide sphere with a steel surface. They point out that better agreement with their data is obtained with a dissipation model that is sublinear in strain-rate. Furthermore, the models of $\eta(v)$ yield different dependencies on size of the impinging sphere with $1 - \eta_{visc} \propto 1/R$ whereas $\eta_{plastic} \propto R^0$. A recent review of simulational schemes [12] for granular materials catalogues several simulation models in which η increases, decreases, or is independent of R. Experiments that winnow down this wide range of choices are currently lacking. Thus, it is our view that inspite of some high-quality experiments, the available data pool do not allow a decisive experimental validation of models for the size or velocity dependence of η.

We have studied normal collisions of a non-rotating sphere on a massive wall while trying to explore a range of material parameters, impact velocities and sphere size. Here we report results for collisions on two surfaces, the first is a surface-ground steel disc $2.5cm$ in thickness and $22cm$ in diameter resting on a $1.25cm$ thick sorbothane pad to damp vibrations in the plate, the second is a granite optical table $120 \times 80 \times 30cm$ in size. The lateral extent of these surfaces is large enough to eliminate end-effects in the impact [13]. We have used steel, brass, aluminium, copper and plastic (delrin) spheres of radius $R = 0.47$cm.

*Also at Physics Department, Mt. Holyoke College, S. Hadley, MA.
†Electronic address: menon@physics.umass.edu

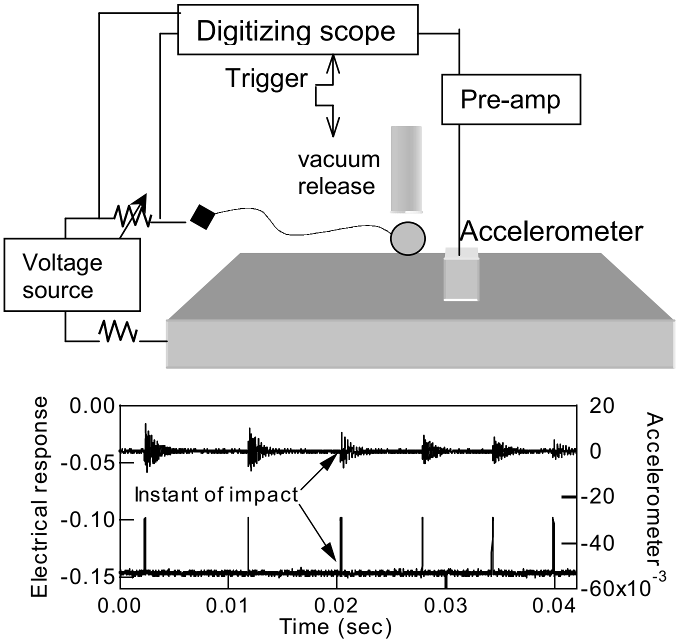

FIG. 1: (A) Schematic of setup, not to scale. (B) Timing traces for a sequence of 6 bounces of a brass ball on a steel plate. The upper trace is from the accelerometer and the lower one from the electrical circuit. Both measurements are delayed with respect to the vacuum release to acquire a short sequence of bounces at a digitization rate sufficient to get ~ 10 points per contact time, t_c.

The brass spheres were varied in size from $R = 0.3125$ to 0.9375cm, a factor of 27 in mass. The ball is held by a vacuum and dropped without spin from a height of about $1cm$ onto a massive plane surface [14] by using a solenoid valve to release the vacuum (see Fig.1). The ball bounces repeatedly on the plane and finally comes to a stop. The voltage pulse that releases the vacuum also triggers acquisition of the times of successive impacts. An accelerometer mounted to the plane about $5cm$ away from the location of the impact detects elastic waves excited by the impact. The instant of the i^{th} impact, t_i, is obtained from the leading edge of the accelerometer pulse (to within a fixed time-lag of $\approx 10\mu s$). When both sphere and plate are metallic, then a second determination of t_i is obtained by applying a small dc voltage between ball and plate and finding the instant when the circuit closes. The electrical method is more sensitive and allows us to measure slower impacts; it also yields the duration of the contact, t_c. We have directly verified that electrostatic forces are negligible in the collision since our results are unchanged when the applied voltage is varied by a factor of 120, or when an ac voltage is used. Where both measurements are possible they yield the same result, as seen in Fig. 1. Given a set of collision times t_i , η at the ith bounce can be determined as $\eta(v_i) = -v_{i+1}/v_i = \frac{t_{i+1}-t_i}{t_i-t_{i-1}}$. In Figure 2 we show the variation of η with impact velocity for a brass sphere on steel. The cloud of small points represents raw data while the solid squares are averages

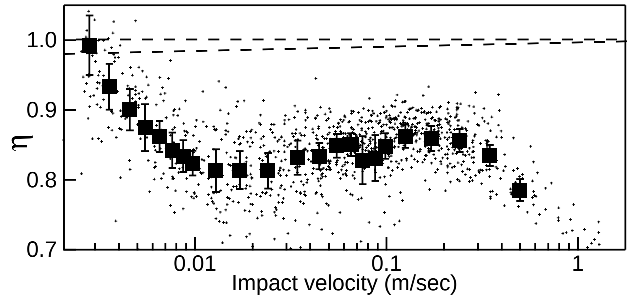

FIG. 2: η versus impact velocity, $v(m/s)$, plotted on a log-scale, for a brass sphere with $R = 0.47cm$ bouncing against a steel plate. The small dots represent 1130 individual collisions, taken over 100 launches of a sphere. The solid squares are averages of these data taken in logarithmically-spaced bins. The error bars are the standard deviation about these values whereas the dashed lines show the precision of individual measurements.

of these data. The novel and striking observation is that the velocity dependence of η is non-monotonic; there is a range of velocities in which the collisions become more elastic as the impact becomes harder. This is at variance with the theories described above which prescribe a monotonic increase towards the limit of $\eta(v = 0) = 1$. It might appear surprising that this non-monotonic dependence has not previously been observed, since our collision geometry is fairly typical and the precision of some previous measurements (e.g. Ref. 11) is comparable to ours. Our understanding of this apparent inconsistency is that resolving a broad and shallow features seen in Fig. 2 requires a much larger number of data points than were taken in previous measurements, and that these data be gathered over a large range in $logv$. The scatter in the raw data of Fig. 2 is not dictated by our precision in measuring η (indicated by the dashed bars in the figure), but by bounce-to-bounce variability due to slight imperfections or asperities falling within the area of contact of these macroscopically smooth objects: only averaging over repeated bounces reveals the underlying behaviour.

In Fig 3 we show that the non-monotonic behaviour is extremely robust and can be seen for impacts between several pairs of materials. Fig 3A shows η as a function of velocity for several metals on a steel plate. While the magnitude of the inelasticity and the position of the minimum and peak in the data vary from one material to another, the overall trends in $\eta(v)$ are maintained. In Fig 3B we show η for collision of a plastic (delrin) sphere on steel and granite surfaces. Once again, $\eta(v)$ clearly shows a peak, though due to the fact that we are not able to use our electrical method of detection, we are unable to go to very small v. The fact that the data are displaced from each other demonstrates the important role of the plate, even when there is a considerable mismatch in the elastic modulus of the materials (delrin being much softer than either steel or granite). Finally, in Fig. 3C, we compare

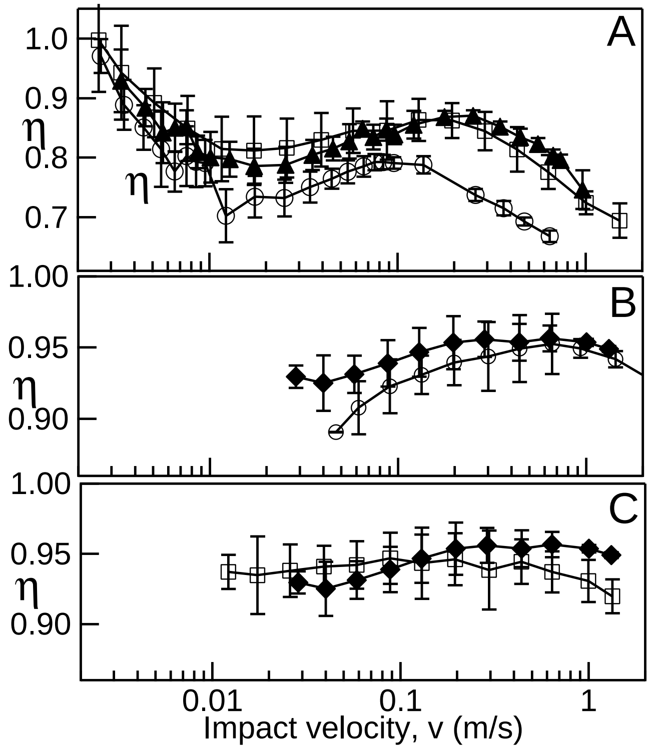

FIG. 3: η versus log v (m/s) for (A) brass (squares), aluminium (▲), and copper (○) spheres bouncing on a steel plate; (B) delrin on granite (♦) and steel (○); and (C) brass (squares) and delrin (♦) bouncing on granite. $R = 0.47cm$ in all these data.

the impact of brass and plastic spheres on the granite surface. We emphasize that the granite plate is much thicker than typical containing walls used in experiments on granular media.

In all the cases above, the lowest impact velocity is determined either by the precision of our technique, or by the ultimate contact of the ball with the plate. (Ref. 11 argues that the last stages of this process are an elastic oscillation of the ball and plate under gravity). The highest velocity we use is determined by the impact speeds at which we first start to observe tiny plastic indentations of ball or plate. When we remain below this velocity there are no *visible* plastic deformations, however, we frequently change spheres and surface-grind the steel plate to guard against ageing. (This does not guarantee that microscopic plastic events do not occur in the collision.) We have tried to eliminate adhesion between sphere and plate by repeatedly cleaning both surfaces between launches of the ball. Our results are unchanged when the experiments are done in a dry N_2 atmosphere and when the sphere and plane are both held at high temperature to expel adsorbed water and volatiles. The viscous drag of the air is also negligible: in the extreme case of a plastic sphere at $v = 100cm/s$, weight/stokes drag $\approx 2 \times 10^4$. Thus we believe we are close to an experimental idealization of the impact problem in which the important forces operative are gravity and the elastic forces of the media. Why then is there such a discrepancy between theory and our observations?

We believe that the answer lies in dynamical effects that occur during the collision. In Fig. 4A we show measurements of the contact time, t_c, of the sphere with the plane which varies with the impact velocity as $t_c \sim v^{-1/5}$, in approximate agreement with the contact time predicted from an elastic, Hertzian collision (as has recently been seen [15] in liquid droplets, where deformations are large and non-Hertzian). In Fig. 4B, we show the velocity of the plate as a function of time with t_c marked on the time-axis for various impact velocities. It is evident that as $t_c(v)$ changes, the phase of motion of the plate at the instant the ball leaves the plate can change substantially. Thus the peak observed in $\eta(v)$ could conceivably be viewed as an elastic mode of the plate slinging the ball upward. The minimum, likewise, could correspond to the plate receding downward at the instant the ball leaves the plate. Since we are not able to measure the acceleration at the location of the impact we do not have a direct verification of this, but the data of Fig. 4 makes this explanation quite compelling.

The effect on η of the crossing of these two time scales – t_c and the vibrational modes of the plate – is likely to be quite generic since the speed of sound in most homogenous solids does not vary by too large a factor. Increasing the thickness of the plates will only shift this crossing

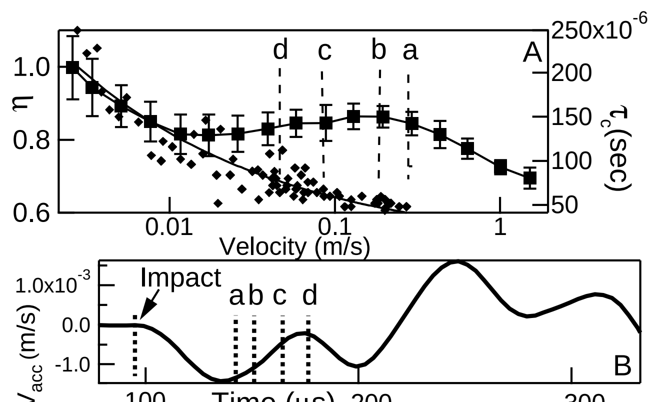

FIG. 4: (A) η (■, left axis) and time of contact, t_c (♦, right axis) versus impact velocity, v. The dashed lines labeled a, b, c, d refer to four individual collisions with impact velocity spaced about the peak in η. (B) Vertical velocity of the plate versus time following the impact. (We show only one of the four traces since they are identical except for amplitude) The instant the ball leaves the plate - as determined by loss of electrical contact - is labeled for each of a,b,c,d. The plate is clearly in a different phase of oscillation for collisions below, above, and at the peak of $\eta(v)$.

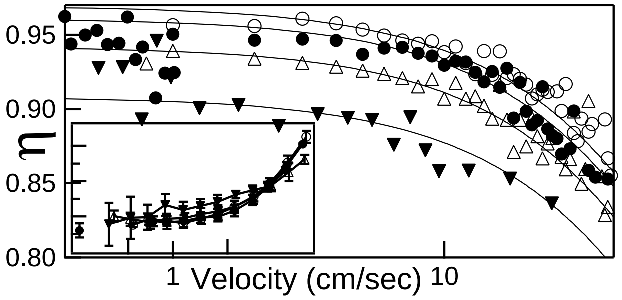

FIG. 5: η vs. impact velocity, v, on a log axis, for the impact on a steel plate of 4 sizes of brass sphere: $R = 0.3125(\bigcirc), 0.47(\bullet), 0.625(\triangle)$, and 0.9375 cm ($\blacktriangledown$). The inset is a plot of $(1-\eta)/R^{1/2}$ vs. v, a scaling that achieves a reasonable collapse of the size dependence.

to lower impact velocities. Furthermore, no matter how massive the plate, it is possible that surface waves on the plate, and modes of the ball, will produce similar effects. The idea that flexural and Rayleigh modes of the plate can play a significant role in the impact is not new [16], however, similar attention has not been paid to bulk modes. Likewise, it has long been known [17] that elastic vibrations can contribute to η even without any further dissipation mechanism. These ideas have been elaborated in recent continuum [18] and lattice [19] simulations of normal impacts of discs against rigid walls.

We have also varied the size of the sphere for brass on steel impacts. We show in Fig. 5 data for η, plotted against v, for 4 different radii of brass spheres. The data show a clear trend for η to decrease as R increases. In the inset to the figure we show that the dependence on radius is consistent with a scaling of $1-\eta \propto R^{1/2}$. The measured dependence is inconsistent with the size dependence of η_{visc}, which displays an increase in η with increase in R, as well as with $\eta_{plastic}$, which is independent of R. These observations, too, appear consistent with vibrations of the plate and sphere being important loss mechanisms, however, a quantitative theory is clearly necessary. Labous et al. [9] state that their data for collisions of nylon spheres shows an increase in η with size consistent with a scaling of $1-\eta \propto R^{-1/2}$, a trend opposite to that shown in Fig 5. The variations in eta for their different sizes, however, are close to the scatter in the data so it difficult to ascertain whether our results are in contradiction.

In conclusion, we have presented data that reveal an unexpectedly complex, nonmonotonic, functional dependence of coefficient of restitution on impact velocity. We have made measurements over a broad range of impact velocity, materials, and particle sizes and find this behaviour to be quite robust. The origin of this nonmonotonic behaviour lies in the fact that the characteristic modes of vibration in the objects participating in a collision are comparable to the contact time in an impact. Experiments are in progress to achieve a situation where these time scales are well-separated. However, in general, we expect dynamical effects to be typical rather than unusual for collisions in granular media and to constrain the regime where quasistatic approximations [2, 3, 4, 5, 6, 7, 8] will apply. Recent work [20, 21] has predicted macroscopic consequences of a velocity-dependent η; it remains to be seen whether there are new consequences that arise from the specific behaviour of $\eta(v)$ that we report. It seems likely that interesting resonant phenomena might occur in sound propagation in granular solids that stem from the velocity dependence we find.

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